2.2. Polyvectors Fields

A polyvector may depend on spacetime points. Let $A = A(x)$ be an r-vector field. Then one can define the *gradient operator* according to

$$\partial = \gamma^\mu \partial_\mu \tag{1}$$

where ∂_μ is the usual partial derivative. The gradient operator ∂ can act on any r-vector field. Using (**??**) we have

$$\partial A = \partial \cdot A + \partial \wedge A \tag{2}$$

Example. Let $A = a = a_\nu \gamma^\nu$ be a 1-vector field. Then

$$\begin{aligned} \partial a &= \gamma^\mu \partial_\mu (a_\nu \gamma^\nu) = \gamma^\mu \cdot \gamma^\nu \, \partial_\mu a^\nu + \gamma^\mu \wedge \gamma^\nu \partial_\mu a_\nu \\ &= \partial_\mu a^\mu + \frac{1}{2}(\partial_\mu a_\nu - \partial_\nu a_\mu) \gamma^\mu \wedge \gamma^\nu \end{aligned} \tag{3}$$

The simple expression ∂a thus contains scalar and bivector part, the former being the usual divergence and the latter the usual curl of a vector field.

Maxwell's equations. We shall demonstrate now on a concrete physical example the usefulness of Clifford algebra. Let us consider the electromagnetic field which, in the language of Clifford algebra, is a bivector field F. The source of the field is the electromagnetic current j which is a 1-vector field. Maxwell's equations read

$$\partial F = 4\pi j \tag{4}$$

The grade of the gradient operator ∂ is 1. Therefore we can use the relation (2) and we find that eq.(4) becomes

$$\partial \cdot F + \partial \wedge F = 4\pi j \tag{5}$$

which is equivalent to

$$\partial \cdot F = -4\pi j \tag{6}$$

$$\partial \wedge F = 0 \tag{7}$$

since the first term on the left of eq.(5) is a vector and the second term is a bivector. This results from the general relation (5). It can also be explicitly demonstrated. Expanding

$$F = \frac{1}{2} F^{\mu\nu} \gamma_\mu \wedge \gamma_\nu \tag{8}$$

$$j = j^\mu \gamma_\mu \tag{9}$$

we have

$$\begin{aligned} \partial \cdot F &= \gamma^\alpha \partial_\alpha \cdot (\frac{1}{2} F^{\mu\nu} \gamma_\mu \wedge \gamma_\nu) = \frac{1}{2} \gamma^\alpha \cdot (\gamma_\mu \wedge \gamma_\nu) \partial_\alpha F^{\mu\nu} \\ &= \frac{1}{2} \left((\gamma^\alpha \cdot \gamma_\mu) \gamma_\nu - (\gamma^\alpha \cdot \gamma_\nu) \gamma_\mu \right) \partial_\alpha F^{\mu\nu} = \partial_\mu F^{\mu\nu} \, \gamma_\nu \end{aligned} \tag{10}$$

$$\partial \wedge F = \frac{1}{2}\gamma^\alpha \wedge \gamma_\mu \wedge \gamma_\nu \, \partial_\alpha F^{\mu\nu} = \frac{1}{2}\epsilon^\alpha{}_{\mu\nu\rho} \, \partial_\alpha F^{\mu\nu} \gamma_5 \gamma^\rho \tag{11}$$

where we have used (**??**) and eqs.(**??**),(**??**). From the above considerations it then follows that the compact equation (4) is equivalent to the usual tensor form of Maxwell's equations

$$\partial_\nu F^{\mu\nu} = -4\pi j^\mu \tag{12}$$

$$\epsilon^\alpha{}_{\mu\nu\rho} \, \partial_\alpha F^{\mu\nu} = 0 \tag{13}$$

Applying the gradient operator ∂ on the left and on the right side of eq.(4) we have

$$\partial^2 F = \partial j \tag{14}$$

Since $\partial^2 = \partial \cdot \partial + \partial \wedge \partial = \partial \cdot \partial$ is a scalar operator, $\partial^2 F$ is a bivector. The right hand side of eq.(14) gives

$$\partial j = \partial \cdot j + \partial \wedge j \tag{15}$$

Equating the terms of the same grade on the left and the right hand side of eq.(14) we obtain

$$\partial^2 F = \partial \wedge j \tag{16}$$

$$\partial \cdot j = 0 \tag{17}$$

The last equation expresses conservation of the electromagnetic current.

Motion of a charged particle. In this example we wish to go a step forward. Our aim is not only to describe how a charged particle moves in an electromagnetic filed, but also include particle' (classical) spin. Therefore, following Pezzaglia [**?**], we define the *momentum polyvector* P as the *vector momentum* p plus the bivector *spin angular momentum* S

$$P = p + S \tag{18}$$

or in components

$$P = p^\mu \gamma_\mu + \frac{1}{2} S^{\mu\nu} \, \gamma_\mu \wedge \gamma_\nu \tag{19}$$

We also assume that the condition $p_\mu S^{\mu\nu} = 0$ is satisfied. The latter condition insures the spin to be a simple bivector, which is purely spacelike in the rest frame of the particle. The polyvector equation of motion is

$$\dot{P} \equiv \frac{\mathrm{d}P}{\mathrm{d}\tau} = \frac{e}{2m} \, [P, F] \tag{20}$$

where $[P, F] \equiv PF - FP$. The vector and bivector parts of eq.(20) are

$$\dot{p}^\mu = \frac{e}{m} \, F^\mu{}_\nu p^\nu \tag{21}$$

$$\dot{S}^{\mu\nu} = \frac{e}{2m} (F^\mu{}_\alpha S^{\alpha\nu} - F^\nu{}_\alpha S^{\alpha\mu}) \tag{22}$$

These are just the equation of motion for linear momentum and spin, respectively.

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