Generalized Dirac equation (Dirac-Kähler equation[1])

$$(i\gamma^{\mu}\partial_{\mu} - m)\Phi = 0$$

$$\Phi = \phi^{A}\gamma_{A} = \psi^{\tilde{A}}\xi_{\tilde{A}} = \psi^{\alpha i}\xi_{\alpha i}$$

Spinor basis of $Cl(1,3)$

α is spinor index of a left minimal ideal.

i runs over four left ideals of $Cl(1,3)$

$$\langle(\xi^{\tilde{A}})^{\ddagger}\gamma^{\mu}\xi_{\tilde{B}}\rangle_{S} \equiv (\gamma^{\mu})^{\tilde{A}}{}_{\tilde{B}}$$

$$\left(i(\gamma^{\mu})^{\tilde{A}}{}_{\tilde{B}}\partial_{\mu} - m\delta^{\tilde{A}}{}_{\tilde{B}}\right)\psi^{\tilde{B}} = 0$$

$$(\gamma^{\mu})^{\tilde{A}}{}_{\tilde{B}} = (\gamma^{\mu})^{\alpha}{}_{\beta}\delta^{i}{}_{j}$$

$$\left(i(\gamma^{\mu})^{\alpha}{}_{\beta}\partial_{\mu} - m\delta^{\alpha}{}_{\beta}\right)\psi^{\beta i} = 0$$

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi^{i} = 0$$

Here we omit spinor index α

Metric in spinor space

$$(\xi_{\tilde{A}})^{\ddagger} * \xi_{\tilde{B}} = z_{\tilde{A}\tilde{B}} = z_{(\alpha i)(\beta j)} = z_{\alpha\beta}z_{ij}$$

$$z_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad z_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Action

$$I = \int \mathrm{d}^4x\, \bar{\psi}^{i}(i\gamma^{\mu}\partial_{\mu} - m)\psi^{j} z_{ij}$$

[1]E. Kähler, Rendiconti di Matematica 21 (1962) 425;
S.I. Kruglov, Dirack-Kähler Equation, arXiv: hep-th/0110251 (and many references therein)
D. Spehled, and G.C. Marques, Eur. Phys. J. 61 (2009) 75